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INTEGRATED RESEARCH ADVANCES

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Certain results on entire functions defined by bicomplex Dirichlet series

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ABSTRACT

In this work, we have introduced and studied the Bicomplex version of Complex Dirichlet Series $f(s)=\sum_{n=1}^{\infty} a_n e^{-ns}$ *. We have derived condition for which the sum function of the Bicomplex Dirichlet Series* $f(\xi)=\sum_{n=1}^{\infty}\alpha_n e^{-n\xi}$ *represents an entire function. The Entireness of sum and Hadamard product of two Entire Bicomplex Dirichlet Series are also discussed..*

Keywords: Dirichlet Series, Entire Dirichlet Series, Riemann Zeta Function, Hadamard Product

INTRODUCTION

Throughout this paper, the set of Bicomplex numbers is denoted by C_2 and the sets of complex and real numbers are denoted by C_1 and C_0, respectively. For details of the theory of bicomplex numbers.[1-3]

The set of Bicomplex Numbers defined as:

$$C_2=\{x_1+i_1x_2+i_2x_3+i_1i_2x_4 : x_1,x_2,x_3,x_4\in C_0,\ i_1\neq i_2 \text{ and } i_1^2=i_2^2=-1,\ i_1i_2=i_2i_1\}$$

We shall use the notations $C(i_1)$ and $C(i_2)$ for the following sets:

$$C(i_1)=\{u+i_1v : u,v\in C_0\}$$

$$C(i_2)=\{\alpha+i_2\beta : \alpha,\beta\in C_0\}$$

1.1 IDEMPOTENT ELEMENTS:

Besides 0 and 1, there are exactly two non – trivial idempotent elements in C_2, denoted as e_1 and e_2 and defined as $e_1=\frac{1+i_1i_2}{2}$ and $e_2=\frac{1-i_1i_2}{2}$

Note that $e_1+e_2=1$ and $e_1e_2=e_2e_1=0$.

1.2 CARTESIAN IDEMPOTENT SET:

Cartesian idempotent set X determined by X_1 and X_2 is denoted as $X_1\times_e X_2$ and is defined as

X=

$$X_1\times_e X_2=\left\{\xi\in C_2 : \xi={}^1\xi e_1+{}^2\xi e_2, \left({}^1\xi, {}^2\xi\right)\in X_1\times X_2\right\}$$

$$C_2=C(i_1)\times_e C(i_1)=C(i_1)e_1+C(i_1)e_2$$
$$=\{\xi\in C_2 : \xi={}^1\xi e_1+{}^2\xi e_2, ({}^1\xi, {}^2\xi)\in C(i_1)\times C(i_1)\}$$

$$C_2=C(i_2)\times_e C(i_2)=C(i_2)e_1+C(i_2)e_2$$
$$=\{\xi\in C_2 : \xi=\xi_1e_1+\xi_2e_2, (\xi_1,\xi_2)\in C(i_2)\times C(i_2)\}$$

1.3 IDEMPOTENT REPRESENTATION OF BICOMPLEX NUMBERS

(I) $C(i_1)$ -idempotent representation of Bicomplex Number is given by

$$\xi=z_1+i_2z_2=(z_1-i_1z_2)e_1+(z_1+i_1z_2)e_2$$
$$={}^1\xi e_1+{}^2\xi e_2$$

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(II) $C(i_2)$-idempotent representation of Bicomplex Number is given by

$$\xi = (x_1 + i_2 x_3) + i_1(x_2 + i_2 x_4) = w_1 + i_1 w_2$$
$$= (w_1 - i_2 w_2)e_1 + (w_1 + i_2 w_2)e_2 = \xi_1 e_1 + \xi_2 e_2$$

Note 1.1: Out of the two idempotent representation, we use $C(i_1)$-idempotent representation. All the results also proved with the help of $C(i_2)$-idempotent representation technique.

The norm in C_2 is defined as

$$\|\xi\| = \left\{|z_1|^2 + |z_2|^2\right\}^{1/2} = \left[\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}\right]^{1/2}$$
$$= \left[x_1^2 + x_2^2 + x_3^2 + x_4^2\right]^{1/2}$$

C_2 becomes a modified Banach algebra, in the sense that

$$\|\xi.\eta\| \le \sqrt{2}\,\|\xi\|\,\|\eta\| \qquad \ldots (1.1)$$

1.4 Complex Dirichlet series:[4-6]

A Dirichlet series is a series of the form $f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ where $\{\lambda_n\}$ is a strictly monotonically increasing and unbounded sequence of positive real numbers, and $s = \sigma + it$ is a complex variable.

When the sequence $\{\lambda_n\}$ of exponent is to be emphasized, such a series is called a complex Dirichlet series of type $\boldsymbol{\lambda_n}$.[4]

A Dirichlet series of the type n is a power series in e^{-s} is given by $f(s) = \sum_{n=1}^{\infty} a_n e^{-ns} = \sum_{n=1}^{\infty} a_n (e^{-s})^n$

A Dirichlet series of type $\mathbf{\log n}$ is the Generalized Riemann Zeta function is given by $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$

Abscissae of convergence and absolute convergence:

To every Dirichlet series, there exists a number σ_0 such that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n e^{-ns}$ converges for $\mathrm{Re}(s) > \sigma_0$ and diverges for $\mathrm{Re}(s) < \sigma_0$. The number σ_0 is called the abscissa of convergence of the series, and the line $\mathrm{Re}(s) = \sigma_0$ is called the line of convergence.

To every Dirichlet series, there exists a number $\bar{\sigma}$ such that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n e^{-ns}$ is absolutely convergent for $\mathrm{Re}(s) > \bar{\sigma}$, and not absolutely convergent for $\mathrm{Re}(s) < \bar{\sigma}$ (this region comprise the region $\mathrm{Re}(s) < \sigma_0$ of divergence, the region $\sigma_0 < \mathrm{Re}(s) < \bar{\sigma}$ of conditional convergence and the line $\mathrm{Re}(s) = \sigma_0$).

The quantity $\bar{\sigma}$ is called the abscissa of absolute convergence of the series $f(s) = \sum_{n=1}^{\infty} a_n e^{-ns}$ and the line $\mathrm{Re}(s) = \bar{\sigma}$ is called the line of absolute convergence.

1.5 Entireness of Complex Dirichlet Series:

Theorem 1.1:[4]

For the complex Dirichlet Series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$

If $\overline{\lim} \frac{n}{\lambda_n} < \infty$, Then $\sigma_0 = \bar{\sigma} = \overline{\lim} \frac{\log|a_n|}{\lambda_n}$.

Corollary 1.1: For a Dirichlet Series $f(s) = \sum_{n=1}^{\infty} a_n e^{-ns}$

$$\sigma_0 = \bar{\sigma} = \overline{\lim} \frac{\log|a_n|}{n}$$

Proof: $\lambda_n = n \Rightarrow \overline{\lim} \frac{n}{\lambda_n} = \overline{\lim} \frac{n}{n} = 1 < \infty$

Hence, $\sigma_0 = \bar{\sigma} = \overline{\lim} \frac{\log|a_n|}{n}$

Corollary 1.2: The Complex Dirichlet Series $f(s) = \sum_{n=1}^{\infty} a_n e^{-ns}$ represents an Entire function iff

$$|a_n|^{1/n} \to 0 \text{ as } n \to \infty.$$

Proof:

For entireness of $f(s) = \sum_{n=1}^{\infty} a_n e^{-ns}$

$$\overline{\lim} \frac{\log|a_n|}{n} = -\infty \Leftrightarrow \lim \frac{\log|a_n|}{n} = -\infty$$
$$\Leftrightarrow \lim \log|a_n|^{1/n} = -\infty \Leftrightarrow \lim |a_n|^{1/n} = 0$$
$$\Leftrightarrow |a_n|^{1/n} \to 0 \text{ as } n \to \infty$$

Hence $f(s)=\sum_{n=1}^{\infty} a_n\, e^{-ns}$ represents an Entire function if $|a_n|^{1/n} \to 0$ as $n \to \infty$.

2. BICOMPLEX DIRICHLET SERIES:

In this paper we discuss a Bicomplex Dirichlet Series of type n, which is a Bicomplex Power Series in $e^{-\xi}$

$$f(\xi)=\sum_{n=1}^{\infty} \alpha_n\, e^{-n\xi}=\sum_{n=1}^{\infty} \alpha_n\,(e^{-\xi})^n$$

where $\{\alpha_n\}$ is a sequence of bicomplex numbers and ξ is a bicomplex variable.

Note that,

$$\sum_{n=1}^{\infty} \alpha_n\, e^{-n\xi}=\left[\sum_{n=1}^{\infty} {}^1\alpha_n\, e^{-n\,{}^1\xi}\right]e_1+\left[\sum_{n=1}^{\infty} {}^2\alpha_n\, e^{-n\,{}^2\xi}\right]e_2$$

$$\Rightarrow f(\xi)={}^1f({}^1\xi)\,e_1+{}^2f({}^2\xi)\,e_2$$

Where, $f(\xi)=\sum_{n=1}^{\infty} \alpha_n\, e^{-n\xi}$ is a Bicomplex Dirichlet Series and ${}^1f({}^1\xi)=\sum_{n=1}^{\infty} {}^1\alpha_n e^{-n\,{}^1\xi}$, ${}^2f({}^2\xi)=\sum_{n=1}^{\infty} {}^2\alpha_n e^{-n\,{}^2\xi}$ are Complex Dirichlet Series.

Throughout, We denote the abscissae of convergence of the Complex Dirichlet series $\sum_{n=1}^{\infty} {}^1\alpha_n e^{-n\,{}^1\xi}$ and $\sum_{n=1}^{\infty} {}^2\alpha_n e^{-n\,{}^2\xi}$ by σ_1 and σ_2 and abscissae of their absolute convergence by $\overline{\sigma}_1$ and $\overline{\sigma}_2$, respectively.

THEOREM 2.1:

For the Bicomplex dirichlet Series $f(\xi)=\sum_{n=1}^{\infty} \alpha_n\, e^{-n\xi}$

$$\sigma_1=\overline{\sigma}_1=\sigma_2=\overline{\sigma}_2=\overline{\lim}\,\frac{\log\|\alpha_n\|}{n}$$

if, $\alpha_{1n}\alpha_{4n}=\alpha_{2n}\alpha_{3n}$

Proof:

$$\alpha_n=\alpha_{1n}+i_1\alpha_{2n}+i_2\alpha_{3n}+i_1i_2\alpha_{4n}$$

$$\alpha_n={}^1\alpha_n\, e_1+{}^2\alpha_n\, e_2$$

Where, ${}^1\alpha_n=(\alpha_{1n}+\alpha_{4n})+i_1(\alpha_{2n}-\alpha_{3n})$ and ${}^2\alpha_n=(\alpha_{1n}-\alpha_{4n})+i_1(\alpha_{2n}+\alpha_{3n})$

$$\left|{}^1\alpha_n\right|=\left|{}^2\alpha_n\right|=\sqrt{(\alpha_{1n})^2+(\alpha_{2n})^2+(\alpha_{3n})^2+(\alpha_{4n})^2}$$

$\because\ \alpha_{1n}\alpha_{4n}=\alpha_{2n}\alpha_{3n}$

Also ,

$$\|\alpha_n\|=\sqrt{(\alpha_{1n})^2+(\alpha_{2n})^2+(\alpha_{3n})^2+(\alpha_{4n})^2}$$

Hence, $\|\alpha_n\|=\left|{}^1\alpha_n\right|=\left|{}^2\alpha_n\right|$ iff $\alpha_{1n}\alpha_{4n}=\alpha_{2n}\alpha_{3n}$

As, $\sigma_1=\overline{\sigma}_1=\overline{\lim}\,\dfrac{\log\left|{}^1\alpha_n\right|}{n}$ and

$$\sigma_2=\overline{\sigma}_2=\overline{\lim}\,\frac{\log\left|{}^2\alpha_n\right|}{n}\qquad \text{[cf. Cor. 1.1]}$$

Hence, $\sigma_1=\overline{\sigma}_1=\sigma_2=\overline{\sigma}_2=\overline{\lim}\,\dfrac{\log\|\alpha_n\|}{n}$

THEOREM 2.2:

The Bicomplex Dirichlet series $\sum_{n=1}^{\infty} \alpha_n\, e^{-n\xi}$ and the k^th derivative defined by $\sum_{n=1}^{\infty}(-n)^k\,\alpha_n\, e^{-n\xi}$ have the same region of convergence.

Proof:

$$\sigma_1=\overline{\sigma}_1=\overline{\lim}\,\frac{\log\left|{}^1\alpha_n\right|}{n} \quad \text{and}$$

$$\sigma_2=\overline{\sigma}_2=\overline{\lim}\,\frac{\log\left|{}^2\alpha_n\right|}{n}$$

Let ρ_1, ρ_2 and $\overline{\rho}_1,\overline{\rho}_2$ are the associated abscissae of convergence and absolute convergence of the Bicomplex Dirichlet series $\sum_{n=1}^{\infty}(-n)^k\,\alpha_n\, e^{-n\xi}$.

The,

$$\rho_1=\overline{\rho}_1=\overline{\lim}\,\frac{\log\left|(-n)^k\,{}^1\alpha_n\right|}{n}$$

$$=\overline{\lim}\,\frac{\log\left|(-n)^k\right|+\log\left|{}^1\alpha_n\right|}{n}$$

$$=\overline{\lim}\,\frac{k\log n+\log\left|{}^1\alpha_n\right|}{n}$$

$$=k\,\overline{\lim}\,\frac{\log n}{n}+\overline{\lim}\,\frac{\log\left|{}^1\alpha_n\right|}{n}$$

$$= k\,\overline{\lim}\frac{\log n}{n} + \overline{\lim}\frac{\log\left|{}^{1}\alpha_n\right|}{n} = 0 + \overline{\lim}\frac{\log\left|{}^{1}\alpha_n\right|}{n}$$

$$= \overline{\lim}\frac{\log\left|{}^{1}\alpha_n\right|}{n}$$

$$= \sigma_1$$

Similarly,

$$\rho_2 = \overline{\rho}_2 = \overline{\lim}\frac{\log\left|(-n)^k\ {}^{2}\alpha_n\right|}{n} = \overline{\lim}\frac{\log\left|{}^{2}\alpha_n\right|}{n}$$

$$= \sigma_2 .$$

THEOREM 2.3:

The Bicomplex Dirichlet series $\sum_{n=1}^{\infty}\alpha_n\, e^{-n\xi}$ and the Bicomplex Dirichlet series $\sum_{n=1}^{\infty}\frac{\alpha_n}{(-n)^k}\, e^{-n\xi}$ obtained after k–times term-by-term integration of $\sum_{n=1}^{\infty}\alpha_n\, e^{-n\xi}$ have the same region of convergence.

Proof:

$$\sigma_1 = \overline{\sigma}_1 = \overline{\lim}\frac{\log\left|{}^{1}\alpha_n\right|}{n} \quad \text{and}$$

$$\sigma_2 = \overline{\sigma}_2 = \overline{\lim}\frac{\log\left|{}^{2}\alpha_n\right|}{n}$$

Let ρ_1, ρ_2 and $\overline{\rho}_1, \overline{\rho}_2$ are the associated abscissae of convergence and absolute convergence of the Bicomplex Dirichlet series $\sum_{n=1}^{\infty}\frac{\alpha_n}{(-n)^k}\, e^{-n\xi}$.

Then, $\rho_1 = \overline{\rho}_1 = \overline{\lim}\dfrac{\log\left|\dfrac{{}^{1}\alpha_n}{(-n)^k}\right|}{n}$

$$= \overline{\lim}\frac{\log\left|{}^{1}\alpha_n\right|}{n} - k\,\overline{\lim}\frac{\log n}{n}$$

$$= \overline{\lim}\frac{\log\left|{}^{1}\alpha_n\right|}{n} - k\,\lim\frac{\log n}{n} = \overline{\lim}\frac{\log\left|{}^{1}\alpha_n\right|}{n} - 0$$

$$= \overline{\lim}\frac{\log\left|{}^{1}\alpha_n\right|}{n}$$

$$= \sigma_1$$

Similarly,

$$\rho_2 = \overline{\rho}_2 = \overline{\lim}\frac{\log\left|\dfrac{{}^{2}\alpha_n}{(-n)^k}\right|}{n} = \overline{\lim}\frac{\log\left|{}^{2}\alpha_n\right|}{n} = \sigma_2 .$$

ENTIRE BICOMPLEX DIRICHLET SERIES

DEFINITION 2.1:

The Bicomplex Dirichlet series $f(\xi) = \sum \alpha_n\, e^{-n\xi}$ is said to be an entire Bicomplex Dirichlet Series if it is convergent in the entire C_2–space.

THEOREM 2.4:

$\left\|\alpha_n\right\|^{\frac{1}{n}} \to 0$ as $n \to \infty$ if and only if $\left|{}^{1}\alpha_n\right|^{\frac{1}{n}} \to 0$ as $n \to \infty$ and $\left|{}^{2}\alpha_n\right|^{\frac{1}{n}} \to 0$ as $n \to \infty$, where $\alpha_n = {}^{1}\alpha_n e_1 + {}^{2}\alpha_n e_2$.

Proof:

Let $\left\|\alpha_n\right\|^{\frac{1}{n}} \to 0$ as $n \to \infty$

Given $\varepsilon > 0$ $\exists\, m \in N$, such that $\left\|\alpha_n\right\|^{\frac{1}{n}} < \varepsilon$ $\forall\, n \geq m$

Now $\forall\, n \geq m$, $\left\|\alpha_n\right\| < \varepsilon^n \Rightarrow$

$$\left[\frac{\left|{}^{1}\alpha_n\right|^2 + \left|{}^{2}\alpha_n\right|^2}{2}\right]^{\frac{1}{2}} < \varepsilon^n$$

$\Rightarrow \left|{}^{1}\alpha_n\right|^2 + \left|{}^{2}\alpha_n\right|^2 < 2\varepsilon^{2n} \Rightarrow \left|{}^{1}\alpha_n\right|^2 < 2\varepsilon^{2n}$ and $\left|{}^{2}\alpha_n\right|^2 < 2\varepsilon^{2n} \Rightarrow \left|{}^{1}\alpha_n\right|^{\frac{1}{n}} \to 0$ and $\left|{}^{2}\alpha_n\right|^{\frac{1}{n}} \to 0$

Conversely let $\left|{}^{1}\alpha_n\right|^{\frac{1}{n}} \to 0$ as $n \to \infty$ and $\left|{}^{2}\alpha_n\right|^{\frac{1}{n}} \to 0$ as $n \to \infty$

i.e. Given $\varepsilon > 0$ $\exists\, m_1, m_2 \in N$

Such that $\left|{}^{1}\alpha_n\right|^{\frac{1}{n}} < \varepsilon$ $\forall\, n \geq m_1$ and $\left|{}^{2}\alpha_n\right|^{\frac{1}{n}} < \varepsilon$ $\forall\, n \geq m_2$

Let $m = \max(m_1, m_2)$

Then $\forall n \geq m$, $\left|{}^1\alpha_n\right|^{\frac{1}{n}} < \varepsilon$ and $\left|{}^2\alpha_n\right|^{\frac{1}{n}} < \varepsilon$

$$\|\alpha_n\| = \left[\frac{\left|{}^1\alpha_n\right|^2 + \left|{}^2\alpha_n\right|^2}{2}\right]^{\frac{1}{2}} \Rightarrow$$

$$2\|\alpha_n\|^2 = \left|{}^1\alpha_n\right|^2 + \left|{}^2\alpha_n\right|^2 \Rightarrow$$

$$2\|\alpha_n\|^2 < \varepsilon^{2n} + \varepsilon^{2n} = 2\,\varepsilon^{2n}$$

$$\Rightarrow \|\alpha_n\|^{\frac{1}{n}} \to 0 \text{ as } n \to \infty$$

THEOREM 2.5:

The Bicomplex Dirichlet series $f(\xi) = \sum \alpha_n e^{-n\xi}$ is an entire Bicomplex Dirichlet series if and only if both ${}^1f({}^1\xi) = \sum {}^1\alpha_n e^{-n\,{}^1\xi}$ and ${}^2f({}^2\xi) = \sum {}^2\alpha_n e^{-n\,{}^2\xi}$ are entire Complex Dirichlet series.

Corollary 2.1: The Bicomplex Dirichlet series $f(\xi) = \sum \alpha_n e^{-n\xi}$ is an entire Bicomplex Dirichlet series if and only if $\overline{\sigma}_1 = -\infty$ and $\overline{\sigma}_2 = -\infty$.

Corollary 2.2: The Bicomplex Dirichlet series $f(\xi) = \sum \alpha_n e^{-n\xi}$ is an entire Bicomplex Dirichlet series if and only if $\left\|{}^1\alpha_n\right\|^{\frac{1}{n}} \to 0$ and $\left\|{}^2\alpha_n\right\|^{\frac{1}{n}} \to 0$.

Corollary 2.3: The Bicomplex Dirichlet series $f(\xi) = \sum \alpha_n e^{-n\xi}$ is an entire Bicomplex Dirichlet series iff $\|\alpha_n\|^{\frac{1}{n}} \to 0$.

THEOREM 2.6:

Let $h(\xi) = \sum_{n=1}^{\infty} (\alpha_n \beta_n) e^{-n\xi}$ be the Hadamard product of $f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-n\xi}$ and $g(\xi) = \sum_{n=1}^{\infty} \beta_n e^{-n\xi}$. If f and g are entire Bicomplex Dirichlet series, then h is also an entire Bicomplex Dirichlet series.

Proof:

$f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-n\xi}$ and $g(\xi) = \sum_{n=1}^{\infty} \beta_n e^{-n\xi}$ are two Entire Bicomplex Dirichlet Series

$\Rightarrow \|\alpha_n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$ and $\|\beta_n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$

i.e. given $\varepsilon > 0\ \exists\ m_1, m_2 \in N$

Such that $\|\alpha_n\|^{\frac{1}{n}} < \varepsilon \quad \forall\ n \geq m_1$ and $\|\beta_n\|^{\frac{1}{n}} < \varepsilon$ $\forall\ n \geq m_2$

Let $m = \max(m_1, m_2)$

Now $\forall\ n \geq m$, $\|\alpha_n\|^{\frac{1}{n}} < \varepsilon$ and $\|\beta_n\|^{\frac{1}{n}} < \varepsilon$

Now, $\qquad \|\alpha_n \beta_n\| \leq \sqrt{2}\,\|\alpha_n\|\,\|\beta_n\| \qquad \Rightarrow$

$$\|\alpha_n \beta_n\|^{\frac{1}{n}} \leq \left[\sqrt{2}\right]^{\frac{1}{n}} \|\alpha_n\|^{\frac{1}{n}} \|\beta_n\|^{\frac{1}{n}} < (2)^{\frac{1}{2n}} (\varepsilon)(\varepsilon)$$

$$= (2)^{\frac{1}{2n}}\, \varepsilon^2$$

$\Rightarrow \|\alpha_n \beta_n\|^{\frac{1}{n}} < (2)^{\frac{1}{2n}}\, \varepsilon^2 \Rightarrow \|\alpha_n \beta_n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$

Hence $h(\xi) = \sum_{n=1}^{\infty} (\alpha_n \beta_n) e^{-n\xi}$ is an entire Bicomplex Dirichlet Series.

THEOREM 2.7:

If $f(\xi) = \sum_{n=1}^{\infty} \alpha_n e^{-n\xi}$ and $g(\xi) = \sum_{n=1}^{\infty} \beta_n e^{-n\xi}$ be two entire Bicomplex Dirichlet series, then the series $\sum_{n=1}^{\infty} (\alpha_n \pm \beta_n) e^{-n\xi}$ is also an entire Bicomplex Dirichlet series.

THEOREM 2.8:

If $\sum_{n=1}^{\infty} \alpha_n e^{-n\xi}$ is an Entire Bicomplex Dirichlet Series, then k^{th} derivative defined by $\sum_{n=1}^{\infty} (-n)^k \alpha_n e^{-n\xi}$ is also an Entire Bicomplex Dirichlet Series.

Proof:

$f(\xi)=\sum_{n=1}^{\infty}\alpha_n e^{-n\xi}$ is an Entire Bicomplex Dirichlet Series

i.e. $\|\alpha_n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$

Now, $\|(-n)^k \alpha_n\|^{\frac{1}{n}} = (n)^{\frac{k}{n}} \|\alpha_n\|^{\frac{1}{n}} \Rightarrow$ $\|(-n)^k \alpha_n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$ $\because (n)^{\frac{k}{n}} \to 0$ as $n \to \infty$

THEOREM 2.9:

If the Bicomplex Dirichlet series $\sum_{n=1}^{\infty}\alpha_n e^{-n\xi}$ is an entire Bicomplex Dirichlet series then the Bicomplex Dirichlet series $\sum_{n=1}^{\infty}\frac{\alpha_n}{(-n)^k} e^{-n\xi}$ obtained after k–times term-by-term integration of $\sum_{n=1}^{\infty}\alpha_n e^{-n\xi}$ is also an entire Bicomplex Dirichlet series.

Proof:

Let $f(\xi)=\sum_{n=1}^{\infty}\alpha_n e^{-n\xi}$ is an entire Bicomplex Dirichlet series i.e. $\|\alpha_n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$

Now, $\left\|\frac{\alpha_n}{(-n)^k}\right\|^{\frac{1}{n}} = \frac{\|\alpha_n\|^{\frac{1}{n}}}{\|(-n)^k\|^{\frac{1}{n}}} = \frac{\|\alpha_n\|^{\frac{1}{n}}}{(n)^{\frac{k}{n}}} \Rightarrow$ $\left\|\frac{\alpha_n}{(-n)^k}\right\|^{\frac{1}{n}} \to 0$ as $n \to \infty$ $\because (n)^{\frac{k}{n}} \to 0$ as $n \to \infty$

$\Rightarrow \sum_{n=1}^{\infty}\frac{\alpha_n}{(-n)^k} e^{-n\xi}$ is an entire Bicomplex Dirichlet series.

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